

EMBEDDING SUBSHIFTS OF FINITE TYPE INTO THE FIBONACCI-DYCK SHIFT

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ABSTRACT. A necessary and sufficient condition is given for the existence of an embedding of an irreducible subshift of finite type into the Fibonacci-Dyck shift

1. INTRODUCTION

Let Σ be a finite alphabet, and let S be the shift on the shift space $\Sigma^{\mathbb{Z}}$,

$$S((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}, \quad (x_i)_{i \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}.$$

An S -invariant closed subset X of $\Sigma^{\mathbb{Z}}$ is called a subshift. For an introduction to the theory of subshifts see [Ki] or [LM]. A word is called admissible for the subshift $X \subset \Sigma^{\mathbb{Z}}$ if it appears in a point of X . A subshift is uniquely determined by its language of admissible word.

Among the first examples of subshifts are the subshifts of finite type. A subshift of finite type is constructed from a finite set \mathcal{F} of words in the alphabet Σ as the subshift that contains the points in $\Sigma^{\mathbb{Z}}$, in which no word in \mathcal{F} appears. Other prototypical examples of subshifts are the Dyck shifts. To recall the construction of the Dyck shifts [Kr1], let $N > 1$, and let

$$\alpha_-(n), \alpha_+(n), \quad 0 \leq n < N,$$

be the generators of the Dyck inverse monoid [NP] \mathcal{D}_N with the relations

$$\alpha_-(n)\alpha_+(m) = \begin{cases} \mathbf{1}, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

The Dyck shifts are defined as the subshifts

$$D_N \subset (\{\alpha_-(n) : 0 \leq n < N\} \cup \{\alpha_+(n) : 0 \leq n < N\})^{\mathbb{Z}}$$

with the admissible words $(\sigma_i)_{1 \leq i \leq I}$, $I \in \mathbb{N}$, of D_N , $N > 1$, given by the condition

$$\prod_{1 \leq i \leq I} \sigma_i \neq 0.$$

In [HI] a necessary and sufficient condition was given for the existence of an embedding of an irreducible subshift of finite type into a Dyck shift. In [HIK] this result was extended to a wider class of target shifts that contains the \mathcal{D}_N -presentations. With the semigroup \mathcal{D}_N^- (\mathcal{D}_N^+) that is freely generated by $\{\alpha_-(n) : 0 \leq n < N\}$ ($\{\alpha_+(n) : 0 \leq n < N\}$), \mathcal{D}_N -presentations can be described as arising from a finite irreducible directed labelled graph with vertex set \mathcal{V} and edge set Σ and a label map

$$\lambda : \Sigma \rightarrow \mathcal{D}_N^- \cup \{\mathbf{1}\} \cup \mathcal{D}_N^+,$$

that extends to directed paths $b = (b_i)_{1 \leq i \leq I}$, $I > 1$, in the directed graph (\mathcal{V}, Σ) by $\lambda(b) = \prod_{1 \leq i \leq I} \lambda(b_i)$. It is required that there exists for $U, W \in \mathcal{V}$, and for $\beta \in \mathcal{D}_N$, a path b from U to W such that $\lambda(b) = \beta$. The \mathcal{D}_N -presentation $X(\mathcal{V}, \Sigma, \lambda)$ is the

subshift with alphabet Σ and with the set of admissible words given by the set of directed finite paths b in the graph $(\mathcal{V}, \Sigma, \lambda)$ such that $\lambda(b) \neq 0$.

For Dyck shifts the notion of a multiplier was introduced in [HI]. A multiplier of the Dyck shift D_N , $N \in \mathbb{N}$, or more generally of a \mathcal{D}_N -presentation [HIK], is an equivalence class of primitive words in the symbols $\alpha(n)$, $0 \leq n \leq N$. Here a word is called primitive if it is not the power of another word, and two primitive words are equivalent, if one is a cyclic permutation of the other. The multipliers are the primitive necklaces of combinatorics [BP, Section 4]. As a notation for a multiplier we use one of its representatives.

We denote the period of a periodic point p of a \mathcal{D}_N -presentation $X(\mathcal{V}, \Sigma, \lambda)$ by $\Pi(p)$. A periodic point $p = (p_i)_{i \in \mathbb{Z}}$ of a \mathcal{D}_N -presentation $X(\mathcal{V}, \Sigma, \lambda)$, and its orbit, are said to be neutral if there exists an $i \in \mathbb{Z}$ such that $\lambda((p_j)_{i \leq j < i + \Pi(p)}) = \mathbf{1}$, and they are said to have negative (positive) multiplier, if there exists an $i \in \mathbb{Z}$ such that $\lambda((p_j)_{i \leq j < i + \Pi(p)}) \in \mathcal{D}_N^-(\mathcal{D}_N^+)$. More precisely, given a multiplier μ , a periodic point $p = (p_i)_{i \in \mathbb{Z}}$ of a \mathcal{D}_N -presentation $X(\mathcal{V}, \Sigma, \lambda)$, and its orbit, are said to have (negative) multiplier μ_- , if there exists an $i \in \mathbb{Z}$ and a representative $(\alpha(n_j))_{1 \leq j \leq J}$ of μ such that $\lambda((p_j)_{i \leq j < i + \Pi(p)})$ is equal to $\prod_{1 \leq j \leq J} \alpha_-(n_j)$, and are said to have (positive) multiplier μ_+ , if there exists an $i \in \mathbb{Z}$ and a representative $(\alpha(n_j))_{1 \leq j \leq J}$ of μ such that $\lambda((p_j)_{i \leq j < i + \Pi(p)})$ is equal to $\prod_{1 \leq j \leq J} \alpha_+(n_j)$. Denote the set of periodic orbits of length n of the \mathcal{D}_N -presentation $X(\mathcal{V}, \Sigma, \lambda)$ with negative (positive) multiplier by $\mathcal{O}_n^-(X(\mathcal{V}, \Sigma, \lambda))$ ($\mathcal{O}_n^+(X(\mathcal{V}, \Sigma, \lambda))$), and denote the set of its periodic orbits of length n with multiplier $\mu_-(\mu_+)$ by $\mathcal{O}_n(\mu_-)(X(\mathcal{V}, \Sigma, \lambda))$ ($\mathcal{O}_n(\mu_+)(X(\mathcal{V}, \Sigma, \lambda))$). The notion of an exceptional multiplier was introduced in [HI]. A multiplier μ of a \mathcal{D}_N -presentation $X(\mathcal{V}, \Sigma, \lambda)$ is said to be exceptional at period $n \in \mathbb{N}$ if

$$\text{card}(\mathcal{O}_n(\mu_-)(X(\mathcal{V}, \Sigma, \lambda)) \cup \mathcal{O}_n(\mu_+)(X(\mathcal{V}, \Sigma, \lambda))) > \max\{\text{card}(\mathcal{O}_n^-(X(\mathcal{V}, \Sigma, \lambda))), \text{card}(\mathcal{O}_n^+(X(\mathcal{V}, \Sigma, \lambda)))\}.$$

The Fibonacci-Dyck shift is the Markov-Dyck shift [M] of the Fibonacci graph. We introduce the Fibonacci graph as the directed graph with vertex set $\{0, 1\}$ and edge set $\{\beta(0), \beta^-, \beta^+(1)\}$: $\beta(0)$ is a loop at vertex 0, the edge $\beta(1)$ goes from vertex 1 to vertex 0, and the edge β from vertex 0 to vertex 1. Let $(\{0, 1\}, \{\beta^-(0), \beta^-, \beta^-(1)\})$ be a copy of the Fibonacci graph, and let $(\{0, 1\}, \{\beta^+(0), \beta^+, \beta^+(1)\})$ be its reversal. The Fibonacci-Dyck shift is given by a \mathcal{D}_2 -presentation, with

$$\mathcal{V} = \{0, 1\}, \quad \Sigma = \{\beta^-(0), \beta^-, \beta^-(1), \beta^+(0), \beta^+, \beta^+(1)\},$$

and

$$\lambda : \Sigma \rightarrow \{\alpha_-(0), \alpha_-(1), \alpha_+(0), \alpha_+(1)\},$$

given by

$$\lambda(\beta^-(0)) = \alpha_-(0), \lambda(\beta^+(0)) = \alpha_+(0),$$

$$\lambda(\beta^-(1)) = \alpha_-(1), \lambda(\beta^+(1)) = \alpha_+(1),$$

$$\lambda(\beta_-) = \lambda(\beta_+) = \mathbf{1}.$$

For this \mathcal{D}_2 -presentation we chose a label map λ that assigns the label $\mathbf{1}$ to the edges β^- and β^+ . These edges β^- and β^+ arise from the splitting of the edge β in the Fibonacci graph. This edge β is contracted to a vertex in the procedure that turns the Fibonacci graph into the 1-vertex graph with two loops, whose graph inverse semigroup is \mathcal{D}_2 . (For this part of the theory see [Kr2, HK]. In [HK, Section 2] it is shown that the Fibonacci-Dyck shift has Property (A), and in [HK, Section 3] it is shown that its associated semigroup is \mathcal{D}_2 . The procedure is also described in [HK, Section 3]). In this paper we consider exclusively the Fibonacci-Dyck shift F .

The label map λ can be written in the form of a matrix with entries in the semigroup ring of \mathcal{D}_2 :

$$\begin{pmatrix} a^-(0) + a^+(0) & \mathbf{1} + a^+(1) \\ \mathbf{1} + a^-(1) & 0 \end{pmatrix}.$$

Taking the adjoint and applying the involution of the semigroup ring of \mathcal{D}_2 to its entries, does not change the matrix. This symmetry property of the matrix makes visible the time reversal ρ of F , that is given by setting

$$\begin{aligned} \chi(\beta^-) &= \beta^+, \chi(\beta^+) = \beta^-, \\ \chi(\beta^-(0)) &= \beta^+(0), \chi(\beta^+(0)) = \beta^-(0), \\ \chi(\beta^-(1)) &= \beta^+(1), \chi(\beta^+(1)) = \beta^-(1), \end{aligned}$$

and

$$\rho((x_i)_{i \in \mathbb{Z}}) = (\chi(x_{-i}))_{i \in \mathbb{Z}}, \quad x \in F.$$

We denote the set of multipliers of F by \mathcal{M} . The time reversal ρ maps the set $\mathcal{O}_n(\mu_-)$ bijectively onto the set $\mathcal{O}_n(\mu_+)$ and we can note a lemma:

Lemma (a).

$$\text{card}(\mathcal{O}_n(\mu_-)) = \text{card}(\mathcal{O}_n(\mu_+)), \quad n \in \mathbb{N}, \mu \in \mathcal{M}.$$

We also note orbit counts of the Fibonacci-Dyck shift for small periods as a lemma:

Lemma (b).

$$\begin{aligned} \text{card}(\mathcal{O}_3(\alpha^-(0))) &= 2, \\ \text{card}(\mathcal{O}_5(\alpha^-(0))) &= 9, \\ \text{card}(\mathcal{O}_5(\alpha^-(0)\alpha^-(1))) &= 4. \\ \\ \text{card}(\mathcal{O}_4(\alpha^-(0))) &= 2, \\ \text{card}(\mathcal{O}_4(\alpha^-(1))) &= 2, \\ \text{card}(\mathcal{O}_6(\alpha^-(0))) &= 11, \\ \text{card}(\mathcal{O}_6(\alpha^-(1))) &= 11. \end{aligned}$$

We note a consequence of Lemma (a) also as a lemma:

Lemma (c). *A multiplier $\mu \in \mathcal{M}$ is exceptional at period $n \in \mathbb{N}$ if and only if*

$$\text{card}(\mathcal{O}_n(\mu_-)) > \text{card}\left(\bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\mu\}} \mathcal{O}_n(\tilde{\mu}_-)\right).$$

The Fibonacci-Dyck shift has the exceptional multiplier $\alpha(0)$, that is exceptional at period one (and in view of Lemma (b) also at periods three, and five), and the exceptional multiplier $\alpha(1)$, that is exceptional at period two. After introducing notation and terminology in section 2, we show in section 3 that the multiplier $\alpha(0)\alpha(1)$ is not exceptional. In section 4, we prove that the remaining multipliers are not exceptional. Based on these results, and on the results of [HIK], we give in section 5 a necessary and sufficient condition for the existence of an embedding of an irreducible subshift of finite type into the Fibonacci-Dyck shift. The multiplier, especially the exceptional multiplier enter here in an essential way. Moreover, we show in section 6, that the multiplier $\alpha(0)$ is exceptional only at periods one, three, and five, and in section 7, that the multiplier $\alpha(1)$ is exceptional only at period two.

We denote the set of periodic points $p \in F$ with smallest period $n \in \mathbb{N}$ by P_n° , and we denote the set of points $p \in P_n^\circ$ with negative multiplier $\mu \in \mathcal{M}$ by $P_n^\circ(\mu_-)$. In the proofs of Sections 3, 4, and 6, 7 we construct for the multiplier μ in question, and for a suitably chosen period $k \in \mathbb{N}$, shift commuting injections

$$\eta_n : P_n^\circ(\mu_-) \rightarrow \bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\mu\}} P_n^\circ(\tilde{\mu}_-), \quad n > k.$$

This we do by first constructing a partition of $P_n^\circ(\mu)$, $n > k$, (some sets of which may be empty), together with a shift commuting injection of each set of the partition into $\bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\mu\}} P_n^\circ(\tilde{\mu}_-)$, where we show injectivity on each set by describing how a point can be reconstructed from its image under η_n . Then we show that the images under η_n of the sets of the partition are disjoint.

2. PRELIMINARIES

We denote the set of admissible words of the Fibonacci-Dyck shift by \mathcal{L} . The empty word we denote by ϵ , and the length of a word we denote by ℓ .

We denote by $\mathcal{C}(0)$ the circular code of words $c = (c_i)_{1 \leq i \leq I} \in \mathcal{L}$, $I > 1$, that begin with the symbol $\beta^-(0)$ and that are such that $\lambda(c) = \mathbf{1}$, and such that for no index J , $1 < J < I$, one has that

$$\prod_{1 \leq j \leq J} \lambda(c_j) = \mathbf{1}.$$

Also we denote by $\mathcal{C}(1)$ the circular code [BPR, Section 7] of words $c = (c_i)_{1 \leq i \leq I} \in \mathcal{L}$, $I > 3$, that begin with the symbol β^- and end with the word $\beta^+(1)\beta^+$, and that are such that $\lambda(c) = \mathbf{1}$, and such that for no index J , $1 < J < I$, one has that $c_J = \beta^+$, and

$$\prod_{1 \leq j \leq J} \lambda(c_j) = \mathbf{1}.$$

We also have the circular codes

$$\mathcal{C} = \mathcal{C}(0) \cup \{\beta^- \beta^+\} \cup \mathcal{C}(1),$$

and

$$\mathcal{C}^\circ(1) = \beta^-(1)\mathcal{C}^*\beta^+(1).$$

Note that

$$\mathcal{C}(0) = \beta^-(0)\mathcal{C}^*\beta^+(0), \quad \mathcal{C}(1) = \beta^-(\mathcal{C}^\circ(1)^* \setminus \{\epsilon\})\beta^+.$$

A bijection

$$\Psi_\circ : \mathcal{C}^\circ(1) \rightarrow \mathcal{C}(0)$$

is given by

$$\Psi_\circ(\beta^-(1)f\beta^+(1)) = \beta^-(0)f\beta^+(0), \quad f \in \mathcal{C}^*.$$

The bijection $\Psi_\circ : \mathcal{C}^\circ(1) \rightarrow \mathcal{C}(0)$ extends to a bijection

$$\Psi : \mathcal{C}^\circ(1)^* \rightarrow \mathcal{C}(0)^*$$

by

$$\Psi((c_k^\circ)_{1 \leq k \leq K}) = ((\Psi_\circ(c_k^\circ))_{1 \leq k \leq K}), \quad c_k^\circ \in \mathcal{C}^\circ(1), 1 \leq k \leq K, \quad K \in \mathbb{N}.$$

We set

$$\mathcal{B}(1) = \beta^-\mathcal{C}^\circ(1)^*\beta^-(1),$$

and we define a bijection

$$\Xi : \mathcal{B}(1) \rightarrow \mathcal{C}(0),$$

by

$$\Xi(\beta^-f^\circ\beta^-(1)) = \beta^-(0)\Psi(f^\circ)\beta^+(0), \quad f^\circ \in \mathcal{C}^\circ(1)^*.$$

We also set

$$\mathcal{B}(0, 0) = \beta^-(0)\mathcal{C}^*\beta^-(0),$$

and we define a bijection

$$\Phi_0 : \mathcal{C}(0) \rightarrow \mathcal{B}(0, 0),$$

by

$$\Phi_0(\beta^-(0)f\beta^+(0)) = \beta^-(0)f\beta^-(0), \quad f \in \mathcal{C}^*.$$

We also set

$$\mathcal{B}(1, 1) = \mathcal{B}(1)\mathcal{C}^*\beta^-\beta^-(1),$$

and we define an bijection

$$\Phi_1 : \mathcal{C}(1) \rightarrow \mathcal{B}(1, 1),$$

by

$$\Phi_1(\beta^-f^\circ\beta^-(1)f\beta^+(1)\beta^+) = \beta^-f^\circ\beta^-(1)f\beta^-\beta^-(1), \quad f^\circ \in \mathcal{C}^\circ(1)^*, f \in \mathcal{C}^*.$$

We set

$$\mathcal{Q}_0 = (\mathcal{C}(0) \cup \{\beta^-\beta^+\})^* \setminus \{\beta^-\beta^+\}^*,$$

and we define an injection

$$\Delta_0 : \mathcal{Q}_0 \rightarrow \mathcal{L}.$$

For this we let $f \in \mathcal{Q}_0$,

$$f = (c_k)_{1 \leq k \leq K}, \quad c_k \in \mathcal{C}(0) \cup \{\beta^-\beta^+\}, 1 \leq k \leq K, \quad K \in \mathbb{N},$$

set

$$k_o(f) = \max\{k \in [1, K] : c_k \in \mathcal{C}(0)\},$$

and set

$$\Delta_0(f) = ((c_k)_{1 \leq k < k_o(f)}, \Phi_0(c_{k_o(f)}), (c_k)_{k_o(f) < k \leq K}).$$

We also set

$$\mathcal{Q}_1 = \mathcal{C}^* \setminus (\mathcal{C}(0) \cup \{\beta^-\beta^+\})^*,$$

and define an injection

$$\Delta_1 : \mathcal{Q}_1 \rightarrow \mathcal{L}$$

For this we let $f \in \mathcal{Q}_1$,

$$f = (c_k)_{1 \leq k \leq K}, \quad c_k \in \mathcal{C}, 1 \leq k \leq K, \quad K > 1,$$

set

$$k_o(f) = \max\{k \in [1, K] : c_k \in \mathcal{C}(1)\},$$

and set

$$\Delta_1(f) = ((c_k)_{1 \leq k < k_o(f)}, \Phi_1(c_{k_o(f)}), (c_k)_{k_o(f) < k \leq K}).$$

We put a linear order on the alphabet of F . The resulting lexicographic order on \mathcal{L} will be used to single out an element of $\mathbb{Z}/n\mathbb{Z}$, when constructing the shift commuting maps

$$\eta_n : P_n^\circ(\mu_-) \rightarrow \bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\mu\}} P_n^\circ(\tilde{\mu}_-), \quad n > k.$$

If a word appears in a point $p \in P_n^\circ$ with its last symbol at index $i \in \mathbb{Z}$ then we say that the word appears at index i . For $p \in F$ we denote by $\mathcal{I}^{(0)}(p)$ ($\mathcal{I}^{(1)}(p)$) the set of indices $i \in \mathbb{Z}$ such that $p_i = \beta_-(0)(\beta_-(1))$ and

$$\lambda(p_{[i, i+k]}) \neq \mathbf{1}, \quad k \in \mathbb{N}.$$

We say that a word appears openly in $p \in F$ if it appears at an index $i \in \mathcal{I}^{(0)}(p) \cup \mathcal{I}^{(1)}(p)$.

For an element γ of the free monoid that is generated by $\alpha(0)$ and $\alpha(1)$ (or by $\alpha_-(0)$ and $\alpha_-(1)$), e.g. for

$$\gamma = \prod_{0 \leq n < N} \alpha(0)^{K(0,n)} \alpha(1)^{K(1,n)}, \quad K(0,n), K(1,n) \in \mathbb{Z}_+, \quad 0 \leq n < N,$$

we use the notation

$$\nu_0(\gamma) = \sum_{0 \leq n < N} K(0,n), \quad \nu_1(\gamma) = \sum_{0 \leq n < N} K(1,n),$$

and, choosing for γ any representative of the multiplier $\mu \in \mathcal{M}$, we set

$$\nu_0(\mu) = \nu_0(\gamma), \quad \nu_1(\mu) = \nu_1(\gamma).$$

A point $p \in P_n^\circ(\mu_-)$, $\mu \in \mathcal{M}$, determines a $\kappa_p \in \mathbb{N}$ by

$$(\nu_0(\lambda(p_{[0,n]})), \nu_1(\lambda(p_{[0,n]}))) = \kappa_p(\nu_0(\mu), \nu_1(\mu)).$$

For

$$b = \beta^- f^\circ \beta^-(1) \in \mathcal{B}(1),$$

we set

$$\Lambda(b) = \ell(f^\circ),$$

and for

$$b = \beta^-(0) f \beta^-(0) \in \mathcal{B}(0,0),$$

we set

$$\Lambda(b) = \ell(f).$$

We also define a subset $\mathcal{D}(1,1)$ of $\mathcal{B}(1)\mathcal{C}^*\mathcal{B}(1)$ by

$$\mathcal{D}(1,1) = \{\beta^- f^{\circ,-} \beta^-(1) f \beta^- f^{\circ,+} \beta^-(1) \in \beta^- \mathcal{C}^\circ(1)^* \beta^-(1) \mathcal{C}^* \beta^-(0) \mathcal{C}^\circ(1)^* \beta^-(1) : \ell(f) \geq \ell(f^{\circ,-}), \ell(f^{\circ,+})\},$$

and a subset $\mathcal{D}(0,1)$ of $\beta^-(0)\mathcal{C}^*\mathcal{B}(1)$ by

$$\mathcal{D}(0,1) = \{\beta^-(0) f \beta^- f^\circ \beta^-(1) \in \beta^-(0) \mathcal{C}^* \beta^- \mathcal{C}^\circ(1)^* \beta^-(1) : \ell(f) \geq \ell(f^\circ)\},$$

as well as a subset $\mathcal{D}(1,0)$ of $\mathcal{B}(1)\mathcal{C}^* \beta^-(0)$ by

$$\mathcal{D}(1,0) = \{\beta^- f^\circ \beta^-(1) f \beta^-(0) \in \beta^- \mathcal{C}^\circ(1)^* \beta^-(1) \mathcal{C}^* \beta^-(0) : \ell(f) \geq \ell(f^\circ)\},$$

and for

$$d = \beta^- f^{\circ,-} \beta^-(1) f \beta^- f^{\circ,+} \beta^-(1) \in \mathcal{D}(1,1),$$

and

$$d = \beta^-(0) f \beta^- f^\circ \beta^-(1) \in \mathcal{D}(0,1),$$

and

$$d = \beta^- f^\circ \beta^-(1) f \beta^-(0) \in \mathcal{D}(1,0),$$

we set

$$\Lambda(d) = \ell(f).$$

For a point $p \in P_n^\circ(F)$ we denote by $\Lambda(p)$ be the maximal value of $\Lambda(d)$ of words

$$d \in \mathcal{B}(0,0) \cup \mathcal{B}(1) \cup \mathcal{D}(1,1) \cup \mathcal{D}(0,1) \cup \mathcal{D}(1,0)$$

that appear openly in p , and we denote by $\mathcal{J}^{(0,0)}(p)(\mathcal{J}^{(1)}(p), \mathcal{J}^{(1,1)}(p), \mathcal{J}^{(0,1)}(p), \mathcal{J}^{(1,0)}(p))$ the set of indices, at which there appears in p openly a word $d \in \mathcal{B}(0,0)$ ($d \in \mathcal{B}(1), d \in \mathcal{D}(1,1), d \in \mathcal{D}(0,1), d \in \mathcal{D}(1,0)$) such that $\Lambda(p) = \Lambda(d)$, and we denote by $\mathcal{J}_\circ^{(0,0)}(p)(\mathcal{J}_\circ^{(1)}(p), \mathcal{J}_\circ^{(1,1)}(p), \mathcal{J}_\circ^{(0,1)}(p), \mathcal{J}_\circ^{(1,0)}(p))$ the set of indices $j_\circ \in \mathcal{J}^{(0,0)}(p)(\mathcal{J}^{(1)}(p), \mathcal{J}^{(1,1)}(p), \mathcal{J}^{(0,1)}(p), \mathcal{J}^{(1,0)}(p))$ such that the word $p_{(j_\circ-n, j_\circ]}$ is lexicographically the smallest one among the words $p_{(j-n, j]}, j \in \mathcal{J}^{(0,0)}(p)(\mathcal{J}^{(1)}(p), \mathcal{J}^{(1,1)}(p), \mathcal{J}^{(0,1)}(p), \mathcal{J}^{(1,0)}(p))$.

3. THE MULTIPLIER $\alpha(0)\alpha(1)$

Lemma 1. *The multiplier $\alpha(0)\alpha(1)$ is not exceptional for the Fibonacci-Dyck shift.*

Proof. By Lemma (b) the multiplier $\alpha(0)\alpha(1)$ is not exceptional for the Fibonacci-Dyck shift for periods three and five.

We construct shift commuting injections

$$\eta_n : P_n^\circ(\alpha_-(0)\alpha_-(1)) \rightarrow \bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\alpha(0)\alpha(1)\}} P_n^\circ(\tilde{\mu}_-), \quad n > 5.$$

Let $m > 2$. Let $P_{2m+1}^{(0)}$ be the set of $p \in P_{2m+1}^\circ(\alpha_-(0)\alpha_-(1))$, such that $\kappa_p = 1$, which means that

$$p_{(i-2m-1, i]} \in \mathcal{C}^* \beta^-(0) \mathcal{C}^* \beta^- \mathcal{C}^\circ(1)^* \beta^-(1), \quad i \in \mathcal{I}^{(1)}(p),$$

and for $p \in P_{2m+1}^{(0)}$ let the words

$$f^-(p), f^+(p) \in \mathcal{C}^*, \quad f^\circ(p) \in \mathcal{C}^\circ(1)^*,$$

be given by writing

$$p_{(i-2m-1, i]} = f^-(p) \beta^-(0) f^+(p) \beta^- f^\circ(p) \beta^-(1), \quad i \in \mathcal{I}^{(1)}(p).$$

We set

$$P_{2m+1}^{[1]} = \{p \in P_{2m+1}^{(0)} : f^+(p) \in \mathcal{Q}_1\}.$$

The shift commuting map η_{2m+1} is to map a point $p \in P_{2m+1}^{[1]}$ to the point $q \in P_{2m+1}^\circ$ that is given by

$$q_{(i-2m-1, i]} = f^-(p) \beta^-(0) \Delta_1(f^+(p)) \beta^- f^\circ(p) \beta^-(1), \quad i \in \mathcal{I}^{(1)}(p).$$

For $q \in \eta_{2m+1}(P_{2m+1}^{[1]})$ one has

$$q_{(i-2m-1, i]} \in \mathcal{C}^* \mathcal{B}(1, 1) \mathcal{C}^* \mathcal{B}(1) \mathcal{C}^* \beta^-(0), \quad i \in \mathcal{I}^{(0)}(q).$$

and with the words

$$g(q), g^-(q) \in \mathcal{C}^*, g^+(q) \in (\mathcal{C}(0) \cup \{\beta^- \beta^+\})^*, \quad b(q) \in \mathcal{B}(1, 1), h(q) \in \mathcal{B}(1),$$

that are given by

$$q_{(i-2m-1, i]} = g^-(q) b(q) g^+(q) h(q) g(q) \beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

the point $p \in P_{2m+1}^{[0]}$ can be reconstructed from its image q under η_{2m+1} as the point in $P_{2m+1}^{(0)}$ that is given by

$$p_{(i-2m-1, i]} = g^-(q) \Phi_1^{-1}(b(q)) g^+(q) h(q) g(q) \beta^-(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$(P.01[1]) \quad \nu_0(\lambda(\eta_{2m+1}(p)_{[0, 2m+1)})) = 1, \quad \nu_1(\lambda(\eta_{2m+1}(p)_{[0, 2m+1)})) = 3,$$

$$p \in P_{2m+1}^{[1]}.$$

We set

$$P_{2m+1}^{[0]} = \{p \in P_{2m+1}^{(0)} : f^+(p) \in \mathcal{Q}_0\}.$$

The shift commuting map η_{2m+1} is to map a point $p \in P_{2m+1}^{[0]}$ to the point $q \in P_{2m+1}^\circ(F)$ that is given by

$$q_{(i-2m-1, i]} = f^-(p) \beta^-(0) \Delta_0(f^+(p)) \beta^- f^\circ(p) \beta^-(1), \quad i \in \mathcal{I}^{(1)}(p).$$

For $q \in \eta_{2m+1}(P_{2m+1}^{[0]})$ one has

$$q_{(i-2m-1, i]} \in \mathcal{C}^* \beta^-(0) \mathcal{B}(0, 0) \mathcal{C}^* \beta^- \mathcal{C}^\circ(1)^* \beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

and with the words

$$\begin{aligned} g(q) &\in \mathcal{C}^*, g^-(q) \in (\mathcal{C}(0) \cup \{\beta^-\beta^+\})^*, g^+(q) \in \{\beta^-\beta^+\})^*, \\ b(q) &\in \mathcal{B}(0,0), g^\circ(q) \in \mathcal{C}^\circ(1)^*, \end{aligned}$$

that are given by

$$q_{(i-2m-1,i]} = g(q)\beta^-(0)g^-(q)b(q)g^+(q)\beta^-g^\circ(q)\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

the point $p \in P_{2m+1}^{[0]}$ can be reconstructed from its image q under η_{2m+1} as the point in $P_{2m+1}^{(0)}$ that is given by

$$p_{(i-2m-1,i]} = g(q)\beta^-(0)g^-(q)\Phi_0^{-1}(b(q))g^+(q)\beta^-g^\circ(q)\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

We note that

$$(P.01[0]) \quad \nu_0(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 3, \quad \nu_1(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 1,$$

$$p \in P_{2m+1}^{[1]}.$$

We set

$$P_{2m+1}^{[\beta]} = P_{2m+1}^{(0)} \setminus (P_{2m+1}^{[1]} \cup P_{2m+1}^{[0]}).$$

The shift commuting map η_{2m+1} is to map a point $p \in P_{2m+1}^{[\beta]}$ to the point $q \in P_{2m+1}^\circ(F)$ that is given by

$$q_{(i-2m-1,i]} = f^-(p)\beta^-(0)f^+(p)\Xi(\beta^-f^\circ(p)\beta^-(1)), \quad i \in \mathcal{I}^{(1)}(p).$$

With words

$$g^{(\beta)}(q) \in \{\beta^-\beta^+\}^*, \quad c(q) \in \mathcal{C}(0), \quad g(q) \in \mathcal{C}^*,$$

that are given by

$$q_{(i-2m-1,i]} = g^{(\beta)}(q)c(q)g(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

a point $p \in P_{2m+1}^{[\beta]}$ can be reconstructed from its image q under η_{2m+1} as the point $p \in P_{2m+1}^{(\circ)}$, that is given by

$$p_{(i-2m-1,i]} = g^{(\beta)}(q)\Xi^{-1}(c(q))g(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$(P.01[\beta]) \quad \nu_0(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 1, \quad \nu_1(\lambda(\eta_{2m+1}(p)_{[0,2m+1)})) = 0,$$

$$p \in P_{2m+1}^{[\beta]}.$$

We set

$$P_{2m}^{(0)} = \emptyset, \quad m > 3,$$

and for $n > 5$ we set

$$P_n^{(1)} = \{p \in P_n^\circ(\alpha_-(0)\alpha_-(1)) \setminus P_n^{(0)} : \mathcal{J}^{(1)}(p) \neq \emptyset\}.$$

The shift commuting map η_n is to map a point $p \in P_n^{(1)}$ to the point $q \in P_n(F)$ that is obtained by replacing in the point p each of the words $b \in \mathcal{B}(1)$ that appear at the indices in $\mathcal{J}_\circ^{(1)}(p)$ by the word $\Xi(b)$.

A point $p \in P_n^{(1)}$ can be reconstructed from its image q under η_n by replacing in q the word $c(q) \in \mathcal{C}(0)$, that is identified as the unique word in \mathcal{C}^* of maximal length that appears in q , by the word $\Xi^{-1}(c(q))$.

We note that

$$(P.01.1) \quad (\nu_0(\lambda(\eta_n(p)_{[0,n)})), \nu_1(\lambda(\eta_n(p)_{[0,n)}))) = (\kappa_p, \kappa_p - 1), \quad p \in P_n^{(1)}.$$

We set

$$P_n^{(0,1)} = \{p \in P_n^\circ(\alpha_-(0)\alpha_-(1)) \setminus (P_n^{(0)} \cup P_n^{(1)}) : \mathcal{J}^{(0,1)}(p) \neq \emptyset\}.$$

The shift commuting map η_n is to map a point $p \in P_n^{(0,1)}$ to the point $q \in P_n$ that is obtained by replacing in the point p the words $b \in \mathcal{B}(1)$ that appear in p at the indices in $\mathcal{J}_o^{(0,1)}(p)$ by the word $\Phi_0(\Xi(b))$.

A point $p \in P_n^{(0,1)}$ can be reconstructed from its image q under η_n by replacing in q the word

$$\beta^-(0)h(q)b(q) \in \beta^-(0)\mathcal{C}^*\mathcal{B}(0,0),$$

whose prefix $\beta^-(0)h(q)\beta^-(0)$ is identified as the unique word in $\mathcal{B}(0,0)$ of maximal length, that appears openly in q , by the word

$$\beta^-(0)h(q)\Xi^{-1}(\Phi_0^{-1}(b(q))).$$

We note that

$$(P.01.01) \quad (\nu_0(\lambda(\eta_n(p)_{[0,n]})), \nu_1(\lambda(\eta_n(p)_{[0,n]}))) = (\kappa_p + 2, \kappa_p - 1), \quad p \in P_n^{(0,1)}.$$

We set

$$P_n^{(1,0)} = P_n^\circ(\alpha^-(0)\alpha^-(1)) \setminus (P_n^{(0)} \cup P_n^{(1)} \cup P_n^{(0,1)}).$$

With the words $b(p) \in \mathcal{B}(1)$, $f(p) \in \mathcal{C}^*$ that are given by writing the word in $\mathcal{D}(1,0)$ that appears at the indices in $\mathcal{J}_o^{(1,0)}(p)$ as $b(p)f(p)\beta^-(0)$, the shift commuting map η_n is to map a point $p \in P_n^{(1,0)}$ to the point $q \in P_n(F)$ that is obtained by replacing in the point p the words in $\mathcal{D}(1,0)$, that appear at the indices in $\mathcal{J}_o^{(1,0)}(p)$ by the word $\Phi_0(\Xi(b(p)))f(p)\beta^-(0)$.

A point $p \in P_n^{(1,0)}$ can be reconstructed from its image q under η_n by replacing in q the word

$$b(q)\beta^-(0)h(q)\beta^-(0) \in \mathcal{B}(0,0)\mathcal{C}^*\beta^-(0),$$

whose suffix $\beta^-(0)h(q)\beta^-(0)$ is identified as the unique word of maximal length in $\mathcal{B}(0,0)$, that appears openly in q , by the word

$$\Xi^{-1}(\Phi_0^{-1}(b(q)))h(q)\beta^-(0).$$

We note that

$$(P.01.10) \quad (\nu_0(\lambda(\eta(p)_{[0,n]})), \nu_1(\lambda(\eta(p)_{[0,n]}))) = (\kappa_p + 2, \kappa_p - 1), \quad p \in \eta_n(P_n^{(1,0)}).$$

We have produced a partition

$$(P.01) \quad P_n^\circ(\alpha(0)\alpha(0)) = P_n^{(0)} \cup P_n^{(1)} \cup P_n^{(0,1)} \cup P_n^{(1,0)}.$$

In points $q \in \eta_n(P_n^{(0,1)})$ the unique word in $\beta^-(0)\mathcal{C}^*\beta^-(0)$ of maximal length that appears openly in q is followed by a word in $\mathcal{C}^*\beta^-(0)$, whereas in points $q \in \eta_n(P_n^{(1,0)})$ the unique word in $\beta^-(0)\mathcal{C}^*\beta^-(0)$ of maximal length that appears openly in q is followed by a word in $\mathcal{C}^*\beta^-$. From this observation and from (P.01[0]), (P.01[1]), (P.01[β]) and (P.01.1), (P.01.01), (P.01.10) it follows that the images under η_n of the sets of the partition (P.01) are disjoint. From (P.01[0]), (P.01[1]), (P.01[β]) and (P.01.1), (P.01.01), (P.01.10) it follows also that

$$\eta_n(P_n^\circ(\alpha_-(0)\alpha_-(1))) \cap P_n^\circ(\alpha_-(0)) = \emptyset.$$

We have shown that

$$\text{card}(\mathcal{O}_n(\alpha_-(0)\alpha_-(1))) \leq \text{card}\left(\bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\alpha(0)\alpha(1)\}} \mathcal{O}_n(\tilde{\mu}^-)\right).$$

Apply Lemma (c). □

4. THE REMAINING MULTIPLIERS

Lemma 2. *Besides the multipliers $\alpha(0)$ and $\alpha(1)$ the Fibonacci-Dyck shift has no exceptional multipliers.*

Proof. Consider a multiplier

$$\mu \notin \{\alpha(0), \alpha(1), \alpha(0)\alpha(1)\},$$

of F . We construct shift commuting injections

$$\eta_n : P_n^\circ(\mu_-) \rightarrow \bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\mu\}} P_n^\circ(\tilde{\mu}_-), \quad n > 4.$$

We set

$$P_n^{(0,0)} = \{p \in P_n^\circ(\mu) : \mathcal{J}^{(0,0)}(p) \neq \emptyset\}.$$

With the word $b(p) \in \mathcal{B}(0,0)$, that appears in p at the indices in $\mathcal{J}_\circ^{(0,0)}(p)$, the shift commuting map η_n is to map a point $p \in P_n^{(0,0)}$ to the point $q \in P_n$, that is obtained by replacing in p the words $b(p)$ in $\mathcal{B}(0,0)$, that appear in p at the indices in $\mathcal{J}_\circ^{(0,0)}(p)$ by the word $\Phi_0^{-1}(b(p))$.

The point p can be reconstructed from its image q under η_n by replacing in q the word $c(q)$, that is identified as the unique word in $\mathcal{C}(0)$ of maximal length, that appears in q , by the word $\Phi_0(c(q))$.

We note that

$$(0.0) \quad (\nu_0(\lambda(\eta_n(p)_{[0,n]}), \nu_1(\lambda(\eta_n(p)_{[0,n]}))) = (\kappa_p \nu_0(\mu) - 2, \kappa_p \nu_1(\mu)), \quad p \in P_n^{(0,0)}.$$

We set

$$P_n^{(1,1)} = \{p \in P_n^\circ(\mu) \setminus P_n^{(0,0)} : \mathcal{J}^{(1,1)}(p) \neq \emptyset\}.$$

With the words

$$b(p) \in \mathcal{B}(1), \quad f(p) \in \mathcal{C}^*, \quad f^\circ(p) \in \mathcal{C}^\circ(1)^*,$$

that are given by writing the word in $\mathcal{D}(1,1)$, that appears in p at the indices in $\mathcal{J}_\circ^{(1,1)}(p)$, as

$$b(p)f(p)\beta^-f^\circ(p)\beta^-(1),$$

the shift commuting map η_n is to map a point $p \in P_n^{(1,1)}$ to the point $q \in P_n(F)$, that is obtained by replacing in p the words in $\mathcal{D}(1,1)$, that appear in p at the indices in $\mathcal{J}_\circ^{(1,1)}(p)$, by the word

$$\Phi_0(\Xi(b(p))f(p)\beta^+(0)\Psi(f^\circ(p))\beta^-(0)).$$

A point $p \in P_n^{(1,1)}$ can be reconstructed from its image q under η_n by replacing in q the word

$$\beta^-(0)h^-(q)\beta^-(0)h(q)\beta^+(0)h^+(q)\beta^-(0),$$

that is identified as the word with the uniquely determined word $\beta^-(0)h(q)\beta^+(0) \in \mathcal{C}(0)$ of maximal length, that appears in q , as infix, a uniquely determined, openly in q appearing word $h^+(q)\beta^-(0) \in \mathcal{C}(0)^*\beta^-(0)$, as suffix, and a uniquely determined word $\beta^-(0)h^-(q) \in \beta^-(0)\mathcal{C}(0)^*$ as prefix, by the word

$$\beta^-\Psi^{-1}(h^-(q))\beta^-(1)h(q)\beta^-\Psi^{-1}(h^+(q))\beta^-(0).$$

We note that

$$(1.1) \quad (\nu_0(\lambda(\eta_n(p)_{[0,n]}), \nu_1(\lambda(\eta_n(p)_{[0,n]}))) = (\kappa_p \nu_0(\mu) + 2, \kappa_p \nu_1(\mu) - 2), \\ p \in P_n^{(1,1)}.$$

We denote by $P_n^{(0,1,0)}$ the set of points in

$$P_n^\circ(\mu) \setminus (P_n^{(0,0)} \cup P_n^{(1,1)})$$

such that $\mathcal{J}^{(1)}(p) \neq \emptyset$, and such that the word in $\mathcal{B}(1)$, that appears at the indices in $\mathcal{J}_\circ^{(1)}(p)$, is preceded in p by a word in $\beta^-(0)\mathcal{C}^*$, and followed in p by a word in $\mathcal{C}^*\beta^-(0)$.

With the word $f(p) \in \mathcal{C}^*$, that is given by writing the openly appearing word in $\mathcal{C}^*\beta^-(0)$, that follows the word $b(p) \in \mathcal{B}(1)$ that appears in p at the indices in $\mathcal{J}_\circ^{(1)}(p)$, as $f(p)\beta^-(0)$, the shift commuting map η_n is to map a point $p \in P_n^{(0,1,0)}$ to the point $q \in P_n(F)$ that is obtained by replacing in the point p the words in $\mathcal{B}(1)$ that appear in p at the indices in $\mathcal{J}_\circ^{(1)}(p)$, together with the openly appearing words in $\mathcal{C}^*\beta^-(0)$, that follow them, by the word $\Xi(b(p))f(p)\beta^+(0)$.

Denoting for a point $p \in P_n^{(0,1,1)}$ by $q' \in F$ the point, that is obtained from its image q under η_n by replacing in q the unique word $c(q) \in \mathcal{C}(0)$ of maximal length, that appears in q , by the word $\Phi(c(q))$, one sees, that the point p can be reconstructed from q by replacing in the point q' the unique word $c(q') \in \mathcal{C}(0)$ of maximal length, that appears in q' , by the word $\Xi^{-1}(c(q'))$.

We note that

$$(0.1.0) \quad (\nu_0(\lambda(\eta_n(p)_{[0,n]})), \nu_1(\lambda(\eta_n(p)_{[0,n]}))) = (\kappa_p \nu_0(\mu) - 2, \kappa_p \nu_1(\mu) - 1), \\ p \in P_n^{(0,1,0)}.$$

We denote by $P_n^{(\bullet,1,1)}$ the set of points p in

$$P_n^\circ(\mu_-) \setminus (P_n^{(0,0)} \cup P_n^{(0,1,0)})$$

such that $\mathcal{J}^{(1)}(p) \neq \emptyset$, and such that the words in $\mathcal{B}(1)$ that appear at the indices in $\mathcal{J}_\circ^{(1)}(p)$ are followed in p by a word in $\mathcal{C}^*\beta^-\mathcal{C}^\circ(1)^*\beta^-(1)$.

The shift commuting map η_n is to map a point $p \in P_n^{(\bullet,1,1)}$ to the point $q \in P_n(F)$ that is obtained by replacing in p the words $b(p) \in \mathcal{B}(1)$, that appear in p at the indices in $\mathcal{J}_\circ^{(1)}(p)$, by the word $\Phi_0(\Xi(b(p)))$.

A point $p \in P_n^{(\bullet,1,1)}$ can be reconstructed from its image q under η_n by replacing in q the word $h(q)$ that is identified as the unique word in $\mathcal{B}(0,0)$ of maximal length that appears in q , by the word $\Xi^{-1}(\Phi_0^{-1}(h(q)))$.

We note that

$$(\bullet.1.1) \quad (\nu_0(\lambda(\eta_n(p)_{[0,n]})), \nu_1(\lambda(\eta_n(p)_{[0,n]}))) = (\kappa_p \nu_0(\mu) + 2, \kappa_p \nu_1(\mu) - 1), \\ p \in P_n^{(\bullet,1,1)}.$$

We denote by $P_n^{(1,1,\bullet)}$ the set of points in

$$P_n^\circ(\mu_-) \setminus (P_n^{(0,0)} \cup P_n^{(0,1,0)} \cup P_n^{(\bullet,1,1)})$$

such that $\mathcal{J}^{(1)}(p) \neq \emptyset$, and such that the word in $\mathcal{B}(1)$ that appears at indices in $\mathcal{J}_\circ^{(1)}(p)$, is preceded in p by a word in $\beta^-\mathcal{C}^\circ(1)^*\beta^-(1)\mathcal{C}^*$.

The shift commuting map η_n is to map a point $p \in P_n^{(1,1,\bullet)}$ to the point $q \in P_n(F)$ that is obtained by replacing in p the words $b(p) \in \mathcal{B}(1)$, that appear in p at the indices in $\mathcal{J}_\circ^{(1)}(p)$, by the word $\Xi(b(p))$.

A point $p \in P_n^{(1,1,\bullet)}$ can be reconstructed from its image q under η_n by replacing in q the word $c(q)$ that is identified as the unique word in $\mathcal{C}(0)$ of maximal length, that appears in q , by the word $\Xi^{-1}(c(q))$.

We note that

$$(1.1.\bullet) \quad (\nu_0(\lambda(\eta_n(p)_{[0,n]})), \nu_1(\lambda(\eta_n(p)_{[0,n]}))) = (\kappa_p \nu_0(\mu), \kappa_p \nu_1(\mu) - 1), \quad p \in P_n^{(1,1,\bullet)}.$$

We set

$$P_n^{(0,1)} = \{p \in P_n^\circ(\mu_-) \setminus (P_n^{(0,0)} \cup P_n^{(0,1,0)} \cup P_n^{(\bullet,1,1)}) \cup P_n^{(1,1,\bullet)} : \mathcal{J}^{(1,0)}(p) \neq \emptyset\}.$$

The shift commuting map η_n is to map a point $p \in P_n^{(0,1)}$ to the point $q \in P_n(F)$, that is obtained by replacing in p the words $b(p) \in \mathcal{B}(1)$ that appear at the indices in $\mathcal{J}_\circ^{(0,1)}(p)$, by the word $\Phi_0(\Xi(b(p)))$.

The point $p \in P_n^{(0,1)}$ can be reconstructed from its image q under η_n by replacing in q the word $\beta^-(0)g(q)\beta^-(0)$ that is identified as the unique word in $\mathcal{B}(0,0)$ of maximal length, that appears in q , by the word $\beta^-(0)g(q)\beta^-$, and the open appearances $h(q)\beta^-(0)$ of a word in $\mathcal{C}(0)^*\beta^-(0)$, that follow in q the word $\beta^-(0)g(q)\beta^-(0)$, by the word $\Psi^{-1}(h(q))\beta^-(1)$.

We note that

$$(0.1) \quad (\nu_0(\lambda(\eta(p)_{[0,n]})), \nu_1(\lambda(\eta(p)_{[0,n]}))) = (\kappa_p \nu_0(\mu) + 2, \kappa_p \nu_1(\mu) - 1), \quad p \in P_n^{(0,1)}.$$

We set

$$P_n^{(1,0)} = P_n^\circ(\mu_-) \setminus (P_n^{(0,0)} \cup P_n^{(0,1,0)} \cup P_n^{(\bullet,1,1)} \cup P_n^{(1,1,\bullet)} \cup P_n^{(0,1)}).$$

With the words

$$b(p) \in \mathcal{B}, \quad f(p) \in \mathcal{C}^*,$$

that are given by writing the word in $\mathcal{D}(1,0)$, that appears at the indices in $\mathcal{J}_\circ^{(1,0)}(p)$, as

$$b(p)f(p)\beta^-(0),$$

the shift commuting map η_n is to map a point $p \in P_n^{(1,0)}$ to the point $q \in P_n(F)$, that is obtained by replacing in p the words $b(p)f(p)\beta^-(0)$, that appear at the indices in $\mathcal{J}_\circ^{(1,0)}(p)$, by the word

$$\Phi_0(\Xi(b(p))f(p)\beta^+(0)).$$

The point $p \in P_n^{(1,0)}$ can be reconstructed from its image q under η_n by replacing in q the word $\beta^-(0)g(q)\beta^+(0)$, that is identified as the unique word in $\mathcal{C}(0)$ of maximal length, that appears in q , by the word $\beta^-(1)g(q)\beta^-(0)$, and the word $\beta^-(0)h(q) \in \beta^-(0)\mathcal{C}^*$ that precedes the word $\beta^-(0)g(q)\beta^+(0)$ in q , by the word $\beta^-\Psi^{-1}(h(q))$.

We note that

$$(1.0) \quad (\nu_0(\lambda(\eta(p)_{[0,n]})), \nu_1(\lambda(\eta(p)_{[0,n]}))) = (\kappa_p \nu_0(\mu), \kappa_p \nu_1(\mu) - 1), \quad p \in P_n^{(1,0)}.$$

We have produced a partition

$$(P) \quad P_n^\circ(\mu_-) = P_n^{(0,0)} \cup P_n^{(0,1,0)} \cup P_n^{(\bullet,1,1)} \cup P_n^{(1,1,\bullet)} \cup P_n^{(0,1)} \cup P_n^{(1,0)}.$$

In a point $q \in \eta_n(P_n^{(\bullet,1,1)})$ the word in $\mathcal{B}(0,0)$ of maximal length, that appears in q , is followed by an open appearance of a word in $\mathcal{C}^*\mathcal{B}(1)$, whereas in the points $q \in \eta_n(P_n^{(0,1)})$ this word is followed by an open appearance of a word in $\mathcal{C}^*\beta^-(0)$. Also, in a point $q \in \eta_n(P_n^{(1,1,\bullet)})$ the word in $\mathcal{C}(0)$ of maximal length, that appears in q , is preceded by a word in $\mathcal{B}(1)\mathcal{C}^*$, whereas in a point $q \in \eta_n(P_n^{(1,0)})$ this word is preceded by a word in $\beta^-(0)\mathcal{C}^*$.

It follows from these observations and from (00) , $(0,1,0)$, $(\bullet,1,1)$, $(1,1,\bullet)$, $(0,1)$, $(1,0)$, that the images under η_n of the elements of the partition (P) are disjoint. From $(0,0)$, $(0,1,0)$, $(\bullet,1,1)$, $(1,1,\bullet)$, $(0,1)$, $(1,0)$, it follows also, that

$$\eta_n(P_n^\circ(\mu_-)) \cap P_n^\circ(\mu_-) = \emptyset.$$

We have shown that

$$\text{card } \mathcal{O}_n(\mu_-) \leq \text{card} \left(\bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\mu^-\}} \mathcal{O}_n(\tilde{\mu}_-) \right),$$

The lemma follows now from Lemma (c) and Lemma 1. \square

5. AN EMBEDDING THEOREM

Let $P_k(\alpha(0))$ denote the set of points of F of period k with multiplier $\alpha(0)$, let $P_k(\alpha(1))$ denote the set of points of F of period k with multiplier $\alpha(1)$, and let $P_k(\mathbf{1})$ denote the set of points F periodic points of period k , that are neutral. Denote by ζ_1 the zeta function of the neutral periodic points of F , by $\zeta_{\alpha(0)}$ the zeta function of the periodic points of F with multiplier $\alpha(0)$, and by $\zeta_{\alpha(1)}$ the zeta function of the periodic points of F with multiplier $\alpha(1)$.

Lemma.

$$\begin{aligned} \liminf_{k \rightarrow \infty} \frac{1}{k} \log \text{card} (P_k(\mathbf{1}) \cup P_k(\alpha(0))) &= \\ \liminf_{k \rightarrow \infty} \frac{1}{k} \log \text{card} (P_k(\mathbf{1}) \cup P_k(\alpha(1))) &= \frac{3}{2} \log 3 - \log 2. \end{aligned}$$

Proof. Set

$$\xi(z) = \frac{2}{\sqrt{3}} \sin\left(\frac{1}{3} \arcsin \frac{3\sqrt{3}}{2} z\right), \quad 0 \leq z \leq \frac{2}{3\sqrt{3}}.$$

By [KM, (4.8)] we have the generating functions

$$(5.1) \quad g_{C^\circ(1)^*}(z) = \frac{\xi(z)}{z},$$

and

$$(5.2) \quad g_{C^*}(z) = \frac{\xi(z)^2}{z^2},$$

and it follows that

$$(5.3) \quad \zeta_1(z) = g_{C^\circ(1)^*}(z) g_{C^*}(z) = \frac{\xi(z)^3}{z^3}.$$

Using the circular code $C^* \beta^-(0)$ one finds from (5.2) that

$$(5.4) \quad \zeta_{\alpha(0)}(z) = \left(\frac{z}{z - \xi(z)^2}\right)^2,$$

and using the circular code $\beta^-(1) C^* \beta^- C^\circ(1)^*$ one finds from (5.1) and (5.2) that

$$(5.5) \quad \zeta_{\alpha(1)}(z) = \left(\frac{z}{z - \xi(z)^3}\right)^2.$$

(E.g. see [P, Section 5] or [KM, Section 2]). The lemma follows from (5.4), (5.5) and (5.3). \square

Let P_k^+ denote the set of points of F of period k with positive multiplier, $k \in \mathbb{N}$. Let \mathcal{O}_k^+ denote the set of orbits of length k of F with positive multiplier, let $\mathcal{O}_k(\alpha(0))$ denote the set of orbits of length k of F with multiplier $\alpha(0)$, let $\mathcal{O}_k(\alpha(1))$ denote the set of orbits of length k of F with multiplier $\alpha(1)$, and let $\mathcal{O}_k(\mathbf{1})$ denote the set of orbits of length k of F , that are neutral, $k \in \mathbb{N}$.

Theorem. *Let Y be an irreducible subshift of finite type. Let $\mathcal{O}_k(Y)$ be its set of periodic orbits of length $k \in \mathbb{N}$, and let h_Y be its entropy. An embedding of Y into the Fibonacci-Dyck shift exists if and only if at least one of the following conditions is satisfied:*

(a)

$$\text{card}(\mathcal{O}_k(Y)) \leq \text{card}(\mathcal{O}_k(\mathbf{1}) \cup \mathcal{O}_k(\alpha(0))), \quad k \in \mathbb{N},$$

and

$$h(Y) < \frac{3}{2} \log 3 - \log 2.$$

(b)

$$\text{card}(\mathcal{O}_k(Y)) \leq \text{card}(\mathcal{O}_k(\mathbf{1}) \cup \mathcal{O}_k(\alpha(1))), \quad k \in \mathbb{N},$$

and

$$h(Y) < \frac{3}{2} \log 3 - \log 2.$$

(c)

$$\text{card}(\mathcal{O}_k(Y)) \leq \text{card}(\mathcal{O}_k(\mathbf{1}) \cup \mathcal{O}_k^+), \quad k \in \mathbb{N},$$

and

$$h(Y) < 3 \log 2 - \log 3.$$

Proof. The theorem results from an application of Theorem 5.8 of [HIK], that uses Lemma (a) and Lemma 1, Lemma 2 and Lemma 3, and that takes into account that only the exceptional multipliers, in this case the multipliers $\alpha(0)$ and $\alpha(1)$, contribute to the possibility of an embedding beyond the case of negative (or positive) multipliers. In Theorem 5.8 of [HIK] the entropy condition in (a) reads

$$h(Y) < \liminf_{k \rightarrow \infty} \frac{1}{k} \log \text{card}(P_k(\mathbf{1}) \cup P_k(\alpha(0))),$$

the entropy condition in (b) reads

$$h(Y) < \liminf_{k \rightarrow \infty} \frac{1}{k} \log \text{card}(P_k(\mathbf{1}) \cup P_k(\alpha(1))),$$

and the entropy condition in (c) reads

$$h(Y) < \liminf_{k \rightarrow \infty} \frac{1}{k} \log \text{card}(P_k(\mathbf{1}) \cup P_k^+).$$

For (a) and (b) apply Lemma 5.1. For (c) note that by Lemma (a) the right-hand side of the inequality is equal to the topological entropy of F , which is known to be $3 \log 2 - \log 3$ [KM, Section 4]. \square

By Lemma (a)

$$\begin{aligned} (5.6) \quad & \text{card}(\mathcal{O}_k^+) = \frac{1}{2} \text{card}(\mathcal{O}_k \setminus \mathcal{O}_k(\mathbf{1})), \\ & \text{card}(\mathcal{O}_k(\alpha(0))) = 2 \text{card}(\mathcal{O}_k(\alpha^-(0))), \\ & \text{card}(\mathcal{O}_k(\alpha(1))) = 2 \text{card}(\mathcal{O}_k(\alpha^-(1))), \quad k \in \mathbb{N}. \end{aligned}$$

Denote the set of points of F of period k by P_k , $k \in \mathbb{N}$, and denote by ζ the zeta function of F . The sequence $(\text{card}(\mathcal{O}_k))_{k \in \mathbb{N}}$ can be obtained by Möbius inversion from the sequence $(\text{card}(P_k))_{k \in \mathbb{N}}$ that enters into the zeta function $\zeta(z) = e^{\sum_{n \in \mathbb{N}} \frac{\text{card}(P_n) z^n}{n}}$ of F . The same applies to the sequences

$$\text{card}(\mathcal{O}_k(\mathbf{1}))_{k \in \mathbb{N}}, \text{card}(P_k(\alpha(0)))_{k \in \mathbb{N}}, \text{card}(\mathcal{O}_k(\alpha(1)))_{k \in \mathbb{N}}.$$

Therefore, by (5.6), the information that is relevant for each of the conditions (a), (b) and (c) of Theorem (5.1) is (in principle) contained in the zeta functions ζ , $\zeta_{\mathbf{1}}$, and $\zeta_{\alpha(0)}$, $\zeta_{\alpha(1)}$. ζ is also known. From [KM, (4.12)]

$$\zeta(z) = \frac{\xi(z)}{z(2\xi(z)^2 + \xi(z) - 1)^2}.$$

We note that the zeta function $\zeta_+(z)$ of the periodic point with positive multiplier is also known: By Lemma (a)

$$\zeta_+(z) = \sqrt{\zeta_{\mathbf{1}}(z)^{-1} \zeta(z)} = \frac{\xi(z)^2}{z^2(2\xi(z)^2 + \xi(z) - 1)}.$$

6. THE MULTIPLIER $\alpha(0)$

Proposition M0. *The multiplier $\alpha(0)$ is exceptional only at periods one, three and five.*

Proof. By Lemma (b) the multiplier $\alpha(0)$ is not exceptional at periods two, four and six. Let

$$(a) \quad n \geq 7.$$

We construct a shift commuting injection

$$\eta_n : P_n^\circ(\alpha_-(0)) \rightarrow \bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\alpha(0)\}} P_n^\circ(\tilde{\mu}_-).$$

Set

$$I(i) = \max\{i^{(0)} \in \mathcal{I}^{(0)} : i^{(0)} < i\}, \quad i \in \mathcal{I}^{(0)}(p), \quad p \in P_n^\circ(\alpha_-(0)).$$

Let $P_n^{(1)}$ be the set of points $p \in P_n^\circ(\alpha_-(0))$ such that the set $\mathcal{I}^{(1)}$ of indices $i^0 \in \mathcal{I}^{(0)}$, such that $p_{(I(i), i)} \in \mathcal{Q}_1$, is not empty.

For $p \in P_n^{(1)}$ we denote by $\mathcal{I}_\circ^{(1)}(p)$ the set of indices $i_\circ^{(1)} \in \mathcal{I}^{(1)}(p)$, such that the word $p_{(I(i_\circ^{(1)}), i_\circ^{(1)})}$ is lexicographically the smallest among the words

$$p_{(I(i^{(1)}), i^{(1)})}, \quad i^{(1)} \in \mathcal{I}^{(1)}(p).$$

With the word $f(p) \in \mathcal{Q}_1$, that is given by writing

$$p_{(I(i_\circ^{(1)}), i_\circ^{(1)})} = f(p)\beta_-(0), \quad i_\circ^{(1)} \in \mathcal{I}_\circ^{(1)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(1)}$ to the point $p \in P_n^\circ(F)$ that is obtained by replacing in the point p the words $f(p)\beta_-(0)$, that appear at the indices in $\mathcal{I}_\circ^{(1)}(p)$ by $\Delta_1(f)\beta_-(0)$.

For $q \in \eta_n(P_n^{(1)})$ there is a unique word $b(q) \in \mathcal{B}(1, 1)$, that appears openly in q , and a point $p \in P_n^{(1)}$ can be reconstructed from from its image q under η_n by replacing in q the word $b(q)$, when it appears openly in q , by the word $\Phi_1^{-1}(b(q))$.

We note that

$$(1) \quad (\nu_0(\lambda(\eta(p)_{[0, n]})), \nu_1(\lambda(\eta(p)_{[0, n]}))) = (\kappa_p, 2), \quad p \in P_n^{(1)}.$$

Let $P_n^{(\beta)}$ be the set of points

$$p \in P_n^\circ(\alpha_-(0)) \setminus P_n^{(1)}$$

such that the set $\mathcal{I}^{(\beta)}(p)$ of indices $i^{(\beta)} \in \mathcal{I}^{(0)}(p)$ at which there appears a word in $\beta^-\beta^+(\mathcal{C}(0)^* \setminus \{\epsilon\})\beta^-(0)$ is not empty.

For $p \in P_n^{(\beta)}$ we denote by $\mathcal{I}_\circ^{(\beta)}(p)$ the set of indices $i_\circ^{(\beta)} \in \mathcal{I}^{(\beta)}(p)$ such that the word $p_{(i_\circ^{(\beta)} - n, i_\circ^{(\beta)})}$ is lexicographically the smallest among the words

$$p_{(i^{(\beta)} - n, i^{(\beta)})}, \quad i^{(\beta)} \in \mathcal{I}^{(\beta)}(p).$$

With the word $f(p) \in \mathcal{C}(0)^*$, that is given by writing

$$p_{(I(i_\circ^{(\beta)}), i_\circ^{(\beta)})} = f(p), \quad i_\circ^{(\beta)} \in \mathcal{I}_\circ^{(\beta)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(\beta)}$ to the point $p \in P_n^\circ(F)$ that is obtained by replacing in the point p the word $\beta^-\beta^+f(p)$, that appears in p at the indices in $\mathcal{I}_\circ^{(\beta)}(p) - 1$, by the word $\beta^-\Psi(f(p))\beta^-(1)$.

For a point $q \in \eta_n(P_n^{(\beta)})$ there is a unique word $b(q) \in \mathcal{B}(1)$, that appears openly in q , and a point $p \in P_n^{(\beta)}$ can be reconstructed from its image q under

η_n by replacing in q the word $b(q)$, when it appears openly in q , by the word $\beta^-\beta^+\Psi^{-1}(b(q))$.

We note that

$$(\beta) \quad (\nu_0(\lambda(\eta(p)_{[0,n]})), \nu_1(\lambda(\eta(p)_{[0,n]}))) = (\kappa_p, 2), \quad p \in P_n^{(\beta)}.$$

Let $P_n^{(\beta,0)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \setminus (P_n^{(1)} \cup P_n^{(\beta)})$$

such that the set $\mathcal{I}^{(\beta,0)}(p)$ of indices $i^{(\beta,0)} \in \mathcal{I}^{(0)}(p)$ at which there appears openly the word $\beta^-\beta^+\beta^-(0)$ is not empty.

For $p \in P_n^{(\beta,0)}$ we denote by $\mathcal{I}_\circ^{(\beta,0)}(p)$ the set of indices $i_\circ^{(\beta,0)} \in \mathcal{I}^{(\beta,0)}(p)$ such that the word $p_{(i_\circ^{(\beta,0)} - n, i_\circ^{(\beta,0)})}$ is lexicographically the smallest among the words

$$p_{(i^{(\beta,0)} - n, i^{(\beta,0)})}, \quad i^{(\beta,0)} \in \mathcal{I}^{(\beta,0)}(p).$$

The shift commuting map η_n is to map a point $p \in P_n^{(\beta,0)}$ to the point $p \in P_n^\circ(F)$ that is obtained by replacing in the point p the word $\beta^-\beta^+$, that appears in p at the indices in $\mathcal{I}_\circ^{(\beta,0)}(p) - 1$, by the word $\beta^-\beta^-(1)$.

In a point of $\eta_n(P_n^{(\beta,0)})$ the word $\beta^-\beta^-(1)$ appears openly, and a point $p \in P_n^{(\beta,0)}$ can be reconstructed from its image q under η_n by replacing in q the word $\beta^-\beta^-(1)$, when it appears openly in q , by the word $\beta^-\beta^+$.

We note that

$$(\beta.0) \quad (\nu_0(\lambda(\eta(p)_{[0,n]})), \nu_1(\lambda(\eta(p)_{[0,n]}))) = (\kappa_p, 2), \quad p \in P_n^{(\beta,0)}.$$

Let $P_n^{(0,2)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \setminus (P_n^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)}),$$

such that

$$\nu_0(p) \geq 2.$$

For $p \in P_n^{(0,2)}$ we denote by $\mathcal{I}_\circ^{(0,2)}(p)$ the set of indices $i \in \mathcal{I}^{(0)}(p)$, such that

$$i - I(i) > 1.$$

For $p \in P_n^{(0,2)}$ we denote by $\mathcal{I}_\circ^{(0,2)}(p)$ the set of indices $i_\circ^{(0,2)} \in \mathcal{I}^{(0,2)}(p)$ such that the word $p_{(i_\circ^{(0,2)} - n, i_\circ^{(0,2)})}$ is lexicographically the smallest among the words

$$p_{(i^{(0,2)} - n, i^{(0,2)})}, \quad i^{(0,2)} \in \mathcal{I}^{(0,2)}(p).$$

With the word $f(p) \in \mathcal{C}(0)^*$, that is given by writing

$$p_{[I(i_\circ^{(0,2)}), i_\circ^{(0,2)}]} = \beta^-(0)f(p)\beta^-(0), \quad i^{(0,2)} \in \mathcal{I}_\circ^{(0,2)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,2)}$ to the point $p \in P_n^\circ(F)$ that is obtained by replacing in the point p the words $\beta^-(0)f(p)\beta^-(0)$, that appear in p at the indices in $\mathcal{I}_\circ^{(0,2)}(p)$ by the word $\beta^-\Psi^{-1}(f(p))\beta^-(1)$.

For a point $q \in \eta_n(P_n^{(0,2)})$ there is a unique word $b(q) \in \mathcal{B}(1)$, that appears openly in q , and a point $p \in P_n^{(0,2)}$ can be reconstructed from its image q under η_n by replacing in q the word $b(q)$, when it appears openly in q , by the word $\beta^-\Psi^{-1}(b(q))\beta^-$.

We note that

$$(0.2) \quad (\nu_0(\lambda(\eta(p)_{[0,n]})), \nu_1(\lambda(\eta(p)_{[0,n]}))) = (\kappa_p - 2, 1), \quad p \in P_n^{(0,2)}.$$

Let $P_n^{(0,1,l)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \setminus (P_n^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)}),$$

such that

$$p_{(i-n,i]} \in \beta^-(0)\beta^+(0)\mathcal{C}(0)^*\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the word $f(p) \in \beta^-(0)\beta^+(0)\mathcal{C}(0)^*\beta^-(0)$, that is given by writing

$$p_{(i-n,i]} = \beta^-(0)\beta^+(0)f(p)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,l)}$ to the point $q \in P_n^\circ(F)$ that is given by

$$q_{(i-n,i]} = \beta^-\beta^-(1)f(p)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1,l)})$ one has

$$q_{(i-n,i]} \in \beta^-\beta^-(1)\mathcal{C}(0)^*\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

and with the word $a(q) \in \mathcal{C}(0)^*$ that is given by writing

$$q_{(i-n,i]} = \beta^-\beta^-(1)a(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

a point $p \in P_n^{(0,1,l)}$ can be reconstructed from its image q under η_n as the point in $P_n^\circ(F)$ that is given by

$$p_{(i-n,i]} = \beta^-(0)\beta^+(0)a(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$(0.1.1) \quad (\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (1, 1), \quad p \in P_n^{(0,1,l)}.$$

Let $P_n^{(0,1,m)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \setminus (P_n^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)}),$$

such that

$$p_{(i-n,i]} \in \beta^-(0)(\mathcal{C}(0)^* \setminus \{\epsilon\})\beta^+(0)(\mathcal{C}(0)^* \setminus \{\epsilon\})\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the words $f(p) \in \mathcal{C}(0)^*$, $g(p) \in \mathcal{C}(0)^*$, that are given by writing

$$p_{(i-n,i]} = \beta^-(0)f(p)\beta^+(0)g(p)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,m)}$ to the point $q \in P_n^\circ(F)$ that is given by

$$q_{(i-n,i]} = \beta^-(1)f(p)\beta^-(0)g(p)\beta^-, \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1,m)})$ one has

$$q_{(i-n,i]} \in \mathcal{C}(0)^*\beta^-(0)\mathcal{C}(0)^*\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

and with the words $a(q) \in \mathcal{C}(0)^*$, $b(q) \in \mathcal{C}^*$, that are given by writing

$$q_{[i,i+n)} = a(q)\beta^-(0)b(q)\beta^-\beta^-(1). \quad i \in \mathcal{I}^{(1)}(q),$$

a point $p \in P_n^{(0,1,0)}$ can be reconstructed from its image q under η_n as the point that is given by

$$p_{(i-n,i]} = a(q)\beta^+(0)b(q)\beta^-(0)\beta^-(0), \quad i \in \mathcal{I}^{(1)}(q).$$

We note that

$$(0.1.m) \quad (\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (1, 1), \quad p \in P_n^{(0,1,0)}.$$

Let $P_n^{(0,1,r)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \setminus (P_n^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)}),$$

such that

$$p_{(i-n,i]} \in \beta^-(0)\beta^-(0)(\mathcal{C}(0)^* \setminus \{\epsilon\})\beta^+(0)\mathcal{C}(0)^*\beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the words $f(p) \in \mathcal{C}(0)^*$, $g(p) \in \mathcal{C}(0)^*$, that are given by writing

$$p_{(i-n, i]} = \beta^-(0)f(p)\beta^+(0)g(p)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,r)}$ to the point $q \in P_n^\circ(F)$ that is given by

$$q_{(i-n, i]} = \beta^-\beta^-(1)f(p)\beta^-(0)g(p)\beta^-(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1,r)})$ one has

$$q_{(i-n, i]} \in \mathcal{C}(0)^*\beta^-(0)\mathcal{C}(0)^*\beta^-(0)\beta^-(0)\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

and with the words $a(q) \in \mathcal{C}(0)^*$, $b(q) \in \mathcal{C}(0)^*$, that are given by writing

$$q_{[i, i+n)} = a(q)\beta^-(0)b(q)\beta^-(0)\beta^-(0)\beta^-\beta^-(1). \quad i \in \mathcal{I}^{(1)}(q),$$

a point $p \in P_n^{(0,1,r)}$ can be reconstructed from its image q under η_n as the point that is given by

$$p_{(i-n, i]} = a(q)\beta^-(0)b(q)\beta^-(0)\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

We note that

$$(0.1.r) \quad (\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (3, 1), \quad p \in P_n^{(0,1,r)}.$$

Let $P_n^{(0,1,r,\beta)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \setminus (P^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)}),$$

such that

$$p_{(i-n, i]} \in \beta^-(0)\beta^-(0)\beta^+(0)(\mathcal{C}(0)^* \setminus \mathcal{C}(0)^*)\beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the word $f(p) \in \mathcal{C}(0)^*$, that is given by writing

$$p_{(i-n, i]} = \beta^-(0)\beta^-(0)\beta^+(0)f(p)\beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,r,\beta)}$ to the point $q \in P_n^\circ(F)$ that is given by

$$q_{(i-n, i]} = \beta^-(0)\beta^-\beta^-(1)f(p)\beta^-(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1,r,\beta)})$ one has

$$q_{(i-n, i]} \in \mathcal{C}(0)^*\beta^-(0)\beta^-(0)\beta^-(0)\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

and with the word $a(q) \in \mathcal{C}(0)^*$ that is given by writing

$$q_{[i, i+n)} = a(q)\beta^-(0)\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(0)}(q),$$

a point $p \in P_n^{(0,1,r,\beta)}$ can be reconstructed from its image q under η_n as the point in $P_n^\circ(F)$ that is given by

$$p_{(i-n, i]} = a(q)\beta^+(0)\beta^-(0)\beta^-(0)\beta^+(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$(0.1.r.\beta) \quad (\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (3, 1), \quad p \in P_n^{(0,1,r,\beta)}.$$

Let $P_n^{(0,1,r,1)}$ be the set of points

$$p \in P_n^\circ(\alpha^-(0)) \setminus (P^{(1)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)}),$$

such that

$$p_{(i-n, i]} \in \beta^-(0)\beta^-(0)\beta^+(0)\mathcal{C}(0)^*\beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p).$$

With the word $f(p) \in \beta^-(0)\beta^+(0)\mathcal{C}(0)^*\beta^-(0)$, that is given by writing

$$p_{(i-n, i]} = \beta^-(0)\beta^-(0)\beta^+(0)f(p)\beta^+(0)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(0,1,r,1)}$ to the point $q \in P_n^\circ(F)$ that is given by

$$q_{(i-n,i]} = \beta^-(1)\beta^-\Psi^{-1}(f(p))\beta^-(1)\beta^-(0)\beta^- \quad i \in \mathcal{I}^{(0)}(p).$$

For a point $q \in \eta_n(P_n^{(0,1,r,1)})$ one has

$$q_{(i-n,i]} \in \beta^-\beta^-(1)\beta^-\mathcal{C}(0)^*\beta^-(1)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

and with the word $a(q) \in \mathcal{C}(0)^*$ that is given by writing

$$q_{(i-n,i]} = \beta^-\beta^-(1)a(q)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q),$$

a point $p \in P_n^{(0,1,r,\beta)}$ can be reconstructed from its image q under η_n as the point in $P_n^\circ(F)$ that is given by

$$p_{(i-n,i]} = \beta^-\beta^-(1)\beta^-\Psi(a(q))\beta^-(1)\beta^-(0), \quad i \in \mathcal{I}^{(0)}(q).$$

We note that

$$(0.1.r.0) \quad (\nu_0(\lambda(\eta(p)_{[0,n)})), \nu_1(\lambda(\eta(p)_{[0,n)}))) = (1, 2), \quad p \in P_n^{(0,1,r,0)}.$$

An inspection of the definition of $P_n^{(0,1,r,1)}$ shows that we have produced a partition

$$(P.0) \quad P_n^\circ(\alpha_-(0)) = P_n^{(1)} \cup P_n^{(\beta,\beta)} \cup P_n^{(\beta)} \cup P_n^{(\beta,0)} \cup P_n^{(0,2)} \cup P_n^{(0,1,l)} \cup P_n^{(0,1,m)} \cup P_n^{(0,1,r)} \cup P_n^{(0,1,r,\beta)} \cup P_n^{(0,1,r,0)}.$$

The points of $\eta_n(P_n^{(1)})$ are the only ones in $\eta_n(P_n^\circ(\alpha_-(0)))$, in which there appears openly a word in $\mathcal{B}(1.1)$. In the points of $\eta_n(P_n^{(\beta)} \cup P_n^{(0,2)})$ the word $\beta^-\beta^-(1)$ does not appear openly, whereas in the words of $\eta_n(P_n^{(0,1,l)} \cup P_n^{(0,1,m)} \cup P_n^{(0,1,r)} \cup P_n^{(0,1,r,\beta)} \cup P_n^{(0,1,r,0)})$ this word does appear openly. In the points of $\eta_n(P_n^{(\beta,0)})$ there appears the word $\beta^-\beta^-(1)\beta^-(0)$ openly, whereas in the In the points of $\eta_n(P_n^{(0,1,l)} \cup P_n^{(0,1,m)} \cup P_n^{(0,1,r)})$ this word does not appear openly. However in the points of $\eta_n(P_n^{(0,1,l)} \cup P_n^{(0,1,m)} \cup P_n^{(0,1,r)})$ the word $\beta^-\beta^-(1)$ does appear openly. The points of $\eta_n(P_n^{(0,1,r,\beta)})$ are the only ones in $\eta_n(P_n^\circ(\alpha_-(0)))$, in which there appears openly a word in $(\mathcal{C}^* \setminus \mathcal{C}(0)^*)\beta^-(0)$, and the points of $\eta_n(P_n^{(0,1,r,1)})$ are the only ones in $\eta_n(P_n^\circ(\alpha_-(0)))$, in which there appears openly a word in $\beta^-\beta^-(1)\mathcal{C}(0)^*\mathcal{B}(1)\mathcal{C}(0)^*$.

From these observations and from (1), (β) , $(\beta,0)$, $(0,2)$, and $(0.1.l)$, $(0.1.m)$, $(0.1.r)$, $(0.1.r,\beta)$, $(0.1.r.0)$ it follows, that the images under η_n of the sets in the partition (P.0) are disjoint. Also $\eta_n(P_n^\circ(\alpha_-(0))) \cap P_n^\circ(\alpha_-(0)) = \emptyset$. We have shown that

$$\text{card } \mathcal{O}_n(\alpha^-(0)) \leq \text{card} \left(\bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\alpha^-(0)\}} \mathcal{O}_n(\tilde{\mu}^-) \right).$$

Apply Lemma (c). □

7. THE MULTIPLIER $\alpha(1)$

Proposition M1. *The multiplier $\alpha(1)$ is exceptional only at period two.*

Proof. We construct shift commuting injections

$$\eta_n : P_n^\circ(\alpha_-(1)) \rightarrow \bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\alpha_-(1)\}} P_n^\circ(\tilde{\mu}^-), \quad n > 2.$$

Let $n > 2$. Denote by $P_n^{(1)}$ the set of $p \in P_n^\circ(\alpha^-(1))$ such that the word $\beta^-\beta^-(1)$ appears openly in p . The shift commuting map η_n is to map a point $p \in P_n^{(1)}$ to the point $q \in P_n(F)$ that is obtained by replacing in p the word $\beta^-\beta^-(1)$, when it appears openly in p , by the word $\beta^-(0)\beta^-(0)$. A point $p \in P_n^{(1)}$ can be

reconstructed from its image q under η_n by replacing in q every word $\beta^-(0)^{2K}$ that appears in q openly, and that is neither preceded nor followed in q by an open appearance of the symbol $\beta^-(0)$, by the word $(\beta^-\beta^-(1))^K$, $K \in \mathbb{N}$.

Denote by $P_n^{(2)}$ the set of

$$p \in P_n^\circ(\alpha^-(1)) \setminus P_n^{(1)}$$

such that words in $\beta^-\mathcal{C}^\circ(1)^*\beta^-(1)$ appear openly in p at least twice during a period. Denote for $p \in P_n^{(2)}$ by $\mathcal{J}(p)$ the set of indices $j \in \mathcal{I}^{(1)}(p)$ such that the word $p_{(n-j,n]}$ is lexicographically the smallest one among the words $p_{(n-i,n]}$, $i \in \mathcal{I}^{(1)}(p)$. With the word $f^\circ(p) \in \mathcal{C}^\circ(1)^*$, that is given by writing the word in $\beta^-(\mathcal{C}^\circ(1)^* \setminus \{\epsilon\})\beta^-(1)$ that appears in p at an index in $\mathcal{J}(p)$ as

$$(a) \quad \beta^-f^\circ(p)\beta^-(1),$$

the shift commuting map η_n is to map a point $p \in P_n^{(2)}$ to the point $q \in P_n(F)$ that is obtained by replacing in the point p each of the words (a) that appear in p at an index in $\mathcal{J}(p)$ by the word

$$\beta^-(0)\Psi(f^\circ(p))\beta^-(0).$$

With the word $f(q) \in \mathcal{C}(0)^*$ such that the word $\beta^-(0)f(q)\beta^-(0)$ appears openly in q , the point p can be reconstructed from its image q under η_n by replacing in q the word

$$\beta^-(0)f(q)\beta^-(0)$$

when it appears openly in q , by the word

$$\beta^-\Psi^{-1}(f(q))\beta^-(1).$$

Denote by $P_n^{(3)}$ the set of

$$p \in P_n^\circ(\alpha^-(1)) \setminus (P_n^{(1)} \cup P_n^{(2)})$$

such that

$$p_{(i-n,i]} \in (\mathcal{C}^* \setminus \{\epsilon\})\beta^-\mathcal{C}^\circ(1)^*\beta^-(1).$$

With the words

$$f(p) \in \mathcal{C}^*, \quad f^\circ(p) \in \mathcal{C}^\circ(1)^*,$$

that are given by writing

$$p_{(i-n,i]} = f(p)\beta^-f^\circ(p)\beta^-(1),$$

and denoting by $\Psi'(c)$ the word that is obtained by removing from the word $\Psi(f^\circ(p))$ its last symbol, the shift commuting map ψ_n is to map a point $p \in P_n^{(3)}$ to the point $q \in P_n^\circ(F)$ that is given by

$$q_{(i-n,i]} = f(p)\beta^-(0)\Psi'(f^\circ(p))\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

With the words

$$f(q) \in \mathcal{C}^* \setminus \{\epsilon\}, \quad f'(q) \in \mathcal{C}(0)^*\beta^-(0)\mathcal{C}(0)^*,$$

that are given by writing

$$q_{(i-n,i]} = f(q)\beta^-(0)f'(q)\beta^-\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q),$$

the point p can be reconstructed from its image q under η_n as the point that is given by

$$p_{(i-n,i]} = f(q)\beta^-\Psi^{-1}(f'(q)\beta^-(0))\beta^-(1), \quad i \in \mathcal{I}^{(1)}(q).$$

Set

$$P_n^{(4)} = P_n^\circ(\alpha^-(1)) \setminus (P_n^{(1)} \cup P_n^{(2)} \cup P_n^{(3)}).$$

One has

$$p_{(i-n,i]} \in \beta^-(1)\beta^-\mathcal{C}^\circ(1)^*, \quad i \in \mathcal{I}^{(1)}(p), \quad p \in P_n^{(4)}.$$

With the words

$$c(p) \in \mathcal{C}^\circ(1), \quad f^\circ(p) \in \mathcal{C}^\circ(1)^*,$$

that are given by writing

$$p_{(i-n,i]} = \beta^-(1)\beta^-c(p)f^\circ(p), \quad i \in \mathcal{I}^{(1)}(p),$$

the shift commuting map η_n is to map a point $p \in P_n^{(4)}$ to the point $q \in P_n(F)$ that is given by

$$q_{(i-n,i]} = \beta^-(1)\Phi_0(\Psi(c(p)))\Psi(f^\circ(p))\beta^-, \quad i \in \mathcal{I}^{(1)}(p).$$

With the words

$$b(q) \in \mathcal{B}(0,0), \quad g(q) \in \mathcal{C}(0)^*,$$

that are given by writing

$$q_{(i-n,i]} = \beta^-(1)b(q)g(q)\beta^-, \quad i \in \mathcal{I}^{(1)}(q),$$

the point p can be reconstructed from its image q under η_n as the point that is given by

$$p_{(i-n,i]} = \beta^-(1)\beta^-\Psi^{-1}(\Phi_0^{-1}(b(q)))\Psi^{-1}(g(q)), \quad i \in \mathcal{I}^{(1)}(q).$$

We have produced a partition

$$(P.1) \quad P_n^\circ(\alpha_-(1)) = \bigcup_{1 \leq l \leq 4} P_n^{(l)}.$$

In the points in $\eta_n(P_n^{(1)})$ there appears openly the word $\beta^-(0)\beta^-(0)$, and the word $\beta^-\beta^-(1)$ does not appear openly, and in the points in $\eta_n(P_n^{(2)})$ neither the word $\beta^-(0)\beta^-(0)$ nor the word $\beta^-\beta^-(1)$ appears openly. In the points in $\eta_n(P_n^{(3)})$ and $\eta_n(P_n^{(4)})$ there appears the word $\beta^-\beta^-(1)$ openly. Also

$$\begin{aligned} (\mathcal{I}^{(1)}(q) + 1) \cap \mathcal{I}^{(0)}(q) &= \emptyset, & q \in \eta_n(P_n^{(3)}), \\ \mathcal{I}^{(1)}(q) + 1 &\subset \mathcal{I}^{(0)}, & q \in \eta_n(P_n^{(4)}). \end{aligned}$$

From these observations it follows that the images under η_n of the sets of the partition (P.1) are disjoint.

We have shown that

$$\text{card}(\mathcal{O}_n(\alpha^-(0))) \leq \text{card}\left(\bigcup_{\tilde{\mu} \in \mathcal{M} \setminus \{\alpha(1)\}} \mathcal{O}_n(\tilde{\mu}_-)\right).$$

Apply Lemma (c). □

REFERENCES

- [BP] J. BERSTEL, D. PERRIN, *The origins of combinatorics on words*, European J. of Combinatorics. **28** (2007), 996 – 1022.
- [BPR] J. BERSTEL, D. PERRIN, CH. REUTENAUER, *Codes and Automata*, Cambridge University Press, Cambridge (2010).
- [HI] T. HAMACHI, K. INOUE, *Embeddings of shifts of finite type into the Dyck shift*, Monatsh. Math. **145** (2005), 107 – 129.
- [HIK] T. HAMACHI, K. INOUE, W. KRIEGER, *Subsystems of finite type and semigroup invariants of subshifts*, J. reine angew. Math. **632** (2009), 37 – 61.
- [HK] T. HAMACHI, W. KRIEGER, *A construction of subshifts and a class of semigroups*, arXiv:1303.4158 [math.DS] (2013)
- [Ki] B. P. KITCHENS, *Symbolic dynamics*, Springer, Berlin, Heidelberg, New York (1998).
- [Kr1] W. KRIEGER, *On the uniqueness of the equilibrium state*, Math. Systems Theory **8** (1974), 97 – 104.
- [Kr2] W. KRIEGER, *On a syntactically defined invariant of symbolic dynamics*, Ergod. Th. & Dynam. Sys. **20** (2000) 501 – 516
- [KM] W. KRIEGER, K. MATSUMOTO, *Zeta functions and topological entropy of the Markov-Dyck shifts*, Münster J. Math. **4** (2011), 171–184.

- [LM] D. LIND AND B. MARCUS, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge (1995).
- [M] K. MATSUMOTO, *C^* -algebras arising from Dyck systems of topological Markov chains*, Math. Scand. **109** (2011), 31 – 54.
- [NP] M. NIVAT AND J.-F. PERROT, *Une généralisation du monoïde bicyclique*, C. R. Acad. Sc. Paris, **271** (1970), pp. 824–827.
- [P] D. PERRIN, *Algebraic combinatorics on words*, Algebraic Combinatorics and Computer Science, H.Crapo and G.-C.Rota, Eds. Springer 2001, 391–430.

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